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The acceleration and diffusion of charged particles in a stochastic magnetic field

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Abstract. The problem of acceleration and diffusion of charged particles in a stochastic magnetic field is investigated. The analysis of earlier authors has been generalised to take into account non-δ-correlated stochastic processes. Using Khasminskii's limit theorem a rigorous analysis of the acceleration and diffusion of charged particles is presented. The results are obtained under very general conditions and are free from many limitations in the earlier work. Stochastic heating of the plasma is discussed briefly.

1. Introduction

The problem of the motion of charged particles in the presence of fluctuating electromagnetic fields has attracted considerable attention in the past decade. This interest is largely due to the importance of the investigation in the explanation of energisation of cosmic rays and stochastic heating of plasmas. The state of the subject up to 1966 has been reviewed in depth by Schatzmann (1968) where he also explained the acceleration mechanism for a rarefied plasma. In such a plasma, since the number of charged particles is small, the mutual interaction between the particles can be ignored and each particle experiences separately the same force. Hall and Sturrock (1967) have analysed the diffusion and acceleration of charged particles in a turbulent magnetic field ignoring collective effects. Newmann and Sturrock (1969) have calculated the electrical conductivity for a rarefied plasma in the presence of a stochastic magnetic field, neglecting the collective behaviour of the particles. These authors, though using a turbulent magnetic field, make approximations whose degree of validity is unknown. This situation is unavoidable as long as one deals with stochastic partial differential equations. Almost all the controlled approximate schemes developed by the leading exponents of stochastic equations, e.g. Papanicalou and Varadhan (1973), are not applicable in these analyses. Therefore, sooner or later, in all the analyses performed so far, δ-correlated processes (giving white noise as the power spectral density) are invoked. In the present paper, a simpler problem is analysed where the external stochastic field is assumed to be spatially uniform but fluctuating randomly with time. Using the limit theorem of Khasminskii (1966), a rigorous analysis of the motion of a single charged particle is investigated.

Consider the motion of charged particles under the influence of an axially symmetric electromagnetic field given by

$$\boldsymbol{B} = B(t)\hat{\boldsymbol{e}}_z$$

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and (1)

$$E = -\tfrac{1}{2}(\boldsymbol{r} \times \hat{\boldsymbol{e}}_z)\frac{\mathrm{d}B(t)}{\mathrm{d}t}$$

where $\boldsymbol{r}$ is the position vector.

The magnetic field $\boldsymbol{B}$ consists of a strong DC field B_0, on which a weak randomly fluctuating field is superposed, i.e.

$$\boldsymbol{B} = B_0(1+\epsilon f(t))\hat{\boldsymbol{e}}_z \tag{2}$$

where ϵ is a dimensionless parameter ($\epsilon \ll 1$) and $f(t)$ is a stationary ergodic random function of time, differentiable in the mean square sense and bounded by unity.

The equation of motion of a charged particle subjected to such an electromagnetic field is given by

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{\epsilon\Omega_0}{2}\frac{\mathrm{d}f(t)}{\mathrm{d}t}(\boldsymbol{r}\times\hat{\boldsymbol{e}}_z) + \Omega_0(1+\epsilon f(t))(\boldsymbol{v}\times\hat{\boldsymbol{e}}_z) \tag{3}$$

where Ω_0 is the cyclotron frequency ($=eB_0/m$). Note that the force is proportional to the position vector $\boldsymbol{r}$. This has important ramifications in what follows.

Introducing a new variable:

$$r_+ = (x+\mathrm{i}y)\exp\left(\frac{\mathrm{i}\Omega_0}{2}\int_0^t (1+\epsilon f(t))\,\mathrm{d}t\right) \tag{4}$$

equation (3) can be transformed into

$$\frac{\mathrm{d}^2 r_+}{\mathrm{d}t^2} + \frac{\Omega_0^2}{4}(1+\epsilon f(t))^2 r_+ = 0. \tag{5}$$

It should be emphasised that the above transformation does not in any way affect the stability of motion of the particle as $f(t)$ is bounded for all t.

For small values of ϵ, equation (5) reduces to

$$\frac{\mathrm{d}^2 r_+}{\mathrm{d}t^2} + \frac{\Omega_0^2}{4}(1+2\epsilon f(t)) r_+ = 0. \tag{6}$$

First a more convenient time scale is introduced and equation (6) is reduced to a non-dimensional form, using the correlation length l defined below.

Let $\langle \epsilon f(t)\rangle = 0$, $\langle \epsilon^2 f(t_1) f(t_2)\rangle = R(t_1-t_2)$ and $l = \int_0^\infty R(\tau)\,\mathrm{d}\tau$. l is called the correlation length and is assumed to be finite. Define a new non-dimensional time parameter:

$$t' = t/l.$$

Then r_+ as a function of t' satisfies

$$\frac{\mathrm{d}^2 r_+}{\mathrm{d}t'^2} + \frac{\Omega_0'^2}{4}(1+2\epsilon\bar{f}(t')) r_+ = 0 \tag{7}$$

where

$$\bar{f}(t') = f(t', l) \qquad \Omega_0' = \Omega_0 l.$$

As ϵ is very small, it is possible to introduce two timescales, one the fast timescale determined by Ω_0', and the other a slow timescale determined by $\epsilon\Omega_0'$. The evolution of r_+ can be split into a rapidly varying part determined by the frequency Ω_0', and a slowly varying amplitude and phase determined by ϵ. With these ideas in mind, equation (7) can be cast into the form (Stratanovich and Romanovskii 1965):

$$\mathrm{d}a_+/\mathrm{d}t' = \epsilon \bar{f}(t') \sin 2\Phi \tag{8}$$

$$\mathrm{d}\theta/\mathrm{d}t' = \epsilon w \bar{f}(t')(1 + \cos 2\Phi) \tag{9}$$

where $r_+ = a_+ \cos \Phi$ and $\Phi = \frac{1}{2}\Omega_0' t' + \theta$. a_+ is the slowly varying amplitude and θ is the slowly varying phase.

These equations have been considered earlier in various contexts and certain conclusions can be drawn from them following Stratanovich's method, *vide* Mittal and Prahalad (1977). r_+ exceeds any value A with unit probability as $t' \to \infty$ if the power spectral density of $f(t')$ at Ω_0 is non-zero. That is, the harmonic component of $f(t')$ at the cyclotron frequency transfers its energy to the gyration of the particle, sending the orbital radius to infinity. While this picture is physically very alluring, such a treatment is basically suspect, as it requires the right-hand sides of equations (8) and (9) to be stochastic processes with independent increments, rendering the solution process $a_+(t)$ and $r_+(t)$ to be Markovian. This is possible if the correlation time of $\bar{f}(t')$ is much smaller than the observation time. Mathematically, this corresponds to replacing $\bar{f}(t')$ or $f(t)$ by a δ-correlated stochastic process, which is in conflict with its differentiability property.

Even if $f(t)$ is not a δ-correlated process, using Khasminskii's theorem (1966), it is possible to show that the solution process $a_+(t)$ is indeed a Markov process and obeys a Kalmogrov forward equation with constant coefficients under very general conditions on the right-hand side of equations (8) and (9). This is done in the following way.

Equations (8) and (9) are of the form:

$$\mathrm{d}z_i/\mathrm{d}t' = \epsilon F_i(z(t'), \bar{f}(t'), t') \tag{10}$$

and are amenable to rigorous treatment, rendered possible by Khasminskii's theorem (1966) given below.

2. Khasminskii's theorem

Let $z(t)$ be a stochastic process with values in $\mathbb{R}^n$, defined by the equation

$$\mathrm{d}z_i(t)/\mathrm{d}t = \epsilon F_i(z(t), f(t), t) \tag{11}$$

where $f(t)$ is a $\mathbb{R}^n$ valued stochastic process and F_i are functions of t and $f(t)$, measurable for fixed z and $\langle F_i(z, f(t), t)\rangle = 0$. Furthermore, for some $0 < c < \infty$,

$$|F_i| < c \qquad \left|\frac{\partial F_i}{\partial z_j}\right| < c \qquad \left|\frac{\partial F_i}{\partial z_j\, \partial z_k}\right| < c$$

uniformly in z, $f(t)$ and t for $i, j, k = 1, \ldots, n$. Also the limits

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \left\langle \sum_{j=1}^{n} \frac{\partial F_i(z, f(s), s)}{\partial z_j} F_j(z, f(\sigma), \sigma) \right\rangle \mathrm{d}\sigma\, \mathrm{d}s = b_i(z)$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \left\langle F_i(z, f(s), s) F_j(z, f(\sigma), \sigma) \right\rangle \mathrm{d}\sigma\, \mathrm{d}s = a_{ij}(z)$$

exist uniformly in t_0 and z. Further, if $f(t)$ is strongly mixing, then on the interval

$0 \leq \epsilon^2 t < \tau_0$ (an arbitrary positive number), the solution process $z^{(\epsilon)}(\epsilon^2 t)$ of equation (11) converges as $\epsilon \to 0$ to a Markov process $z^0(\epsilon^2 t)$ which is continuous with probability 1. This has the generator given by

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial z_i \, \partial z_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial z_j}. \tag{12}$$

Note that the right-hand sides of equations (8) and (9) satisfy all the conditions of the theorem. Therefore, it is possible to calculate the generator of the stochastic processes $a_+(t')$ and $\theta(t')$ which, after a straightforward calculation, leads to

$$L = \frac{b}{2} \frac{\partial^2}{\partial a_+^2} + b \frac{\partial}{\partial a_+} + \left(a + \frac{b}{2}\right) \frac{\partial^2}{\partial \theta^2} + c \frac{\partial}{\partial \theta} \tag{13}$$

where

$$a = \frac{\Omega_0'^2}{16} S(0) \qquad b = \frac{\Omega_0'}{16} \operatorname{Re} S(\Omega_0') \qquad c = \frac{\Omega_0'^2}{16} \operatorname{Im} S(\Omega_0')$$

$$S(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\sigma) \exp(-\mathrm{i} w \sigma) \, \mathrm{d}\sigma.$$

$R(\sigma)$ is the correlation function of $\bar{f}(t')$.

The probabilistic properties of $a_+(t')$ and $\theta_+(t')$ can be inferred from the distribution function $P(a_+, \theta, t)$ which is the solution to the initial value problem:

$$\partial P / \partial t = LP \qquad P(0, a_+, \theta, a_{+_0}, \theta_0) = \delta(a_+ - a_{+_0}) \delta(\theta - \theta_0). \tag{14}$$

Solving this equation and transforming back to original variables, the first and second moments of r_+ can be calculated. These are

$$\langle r_+(t') \rangle = \exp\left(\frac{\Omega_0^2}{4} \operatorname{Re} S(\Omega_0) - S(0)\right) \epsilon^2 t' \cos\left[\left(\frac{\Omega_0}{2} - \frac{\epsilon^2 \Omega_0^2}{4} \operatorname{Im} S(\Omega_0)\right) t'\right] \tag{15}$$

i.e. on the slow timescale, the orbits on the average seem to shrink. Further the cyclotron frequency is renormalised by the presence of the field fluctuations. This shift in the frequency is given by $\frac{1}{4}\epsilon^2 \Omega_0^2 \operatorname{Im} S(\Omega_0)$. This frequency renormalisation and the decay of the first moment is the typical effect of fluctuations and occurs in many situations.

Differentiating equation (15) with respect to time, it is at once evident that the velocity $\langle \mathrm{d} r_+ / \mathrm{d} t \rangle$ also decays exponentially. Further differentiation shows that, on the average, $\langle \mathrm{d}^2 r_+ / \mathrm{d} t^2 \rangle$ also decreases, i.e. there is no mean acceleration of the particle.

Computing the second moment, giving the variance of $r_+(t)$, one gets

$$\langle r_+(t)^2 \rangle = \tfrac{1}{2} \exp\left[\frac{\Omega_0^2}{2} (\operatorname{Re} S(\Omega_0) - 2S(0)) \epsilon^2 t\right] \cos\left[\left(\Omega_0 - \frac{\epsilon^2 \Omega_0^2}{2} \operatorname{Im} S(\Omega_0)\right) t\right]$$
$$+ \tfrac{1}{2} \exp[\Omega_0^2 \operatorname{Re} S(\Omega_0) \epsilon^2 t] + \ldots . \tag{16}$$

This shows that the variance grows exponentially on the slow timescale, with a growth rate determined by the power spectral density at the cyclotron frequency. From equation (16), the velocity second moment can be calculated by differentiating twice with respect to time. This immediately shows that the velocity second moment grows exponentially with time. As the kinetic energy is proportional to this moment, it can be seen that the average (executed over an ensemble of realisations of the magnetic field fluctuations or over times large compared to the correlation period) energy of the particle increases exponentially.

It should be noted that these results are only valid asymptotically and indicate a possible mechanism of energisation of charged particles by a stochastic magnetic field, both in the laboratory and in space. Since the collective effects have not been included in this analysis, this study should be helpful to understand the acceleration mechanism of charged particles when their number density is low enough.

If one can also take the collective nature of the plasma into account, noise heating of the plasma for other applications can be seriously considered. The effects are being investigated.

The diffusion equation (14) shows that in addition to velocity space diffusion, the configuration space diffusion also occurs in any realistic treatment. This point is totally lost if one ignores the induced electric field. In the present model, such an approximation will show an average decrease of the velocity of the charged particle and no change in its average energy. This should be hardly surprising. These results are, in a sense, the best possible, at least from a mathematical point of view, as Khasminskii's theorem is a natural generalisation of the central limit theorem of probability theory. It is hoped that the analysis presented above will regenerate the interest in stochastic energisation of particles in a more general context.

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